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Topological Games and Hyperspace Topologies

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Abstract. The paper proposes a unified description of hypertopologies, i.e. topologies on the nonempty closed subsets of a topological space, based on the notion of approach spaces introduced by R. Lowen. As a special case of this description we obtain the abstract hit-and-miss, proximal hit-and-miss and a big portion of weak hypertopologies generated by gap and excess functionals (including the Wijsman topology and the finite Hausdorff topology), respectively. We also give characterizations of separation axioms T_0, T_1, T_2, T_3 and complete regularity as well as of metrizability of hypertopologies in this general setting requiring no additional conditions. All this is done to provide a background for proving the main results in Section 4, where we apply topological games (the Banach–Mazur and the strong Choquet game, respectively) to establish various properties of hypertopologies; in particular we characterize Polishness of hypertopologies in this general setting.

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0. Introduction

There has been an impressive growth in the number of recently introduced topologies on the *hyperspace* (i.e. the collection of all nonempty closed subsets) $\mathrm{CL}(X)$ of a topological space (X, τ) (see [3] for a survey). This increasing interest is owing to usefulness of these so-called *hypertopologies* in different fields of application (such as probability, statistics or variational problems, for instance). It also explains the effort in understanding their structure, common features and general patterns in order to find a common description for them. The papers [4, 41, 42] or more recently [35], are partially or completely devoted to this goal, offering various possibilities of generalization. Let us also mention the paper [34], where the notion of *approach spaces* was first applied in the hyperspace setting to provide a new description of some important hyperspace topologies. It is one of the purposes of our paper to give another description of hyperspace topologies based on approach spaces (cf. [31, 32]). In order to explain the nature of our description, let us first describe the topologies we are going to deal with: for $E \subset X$ write $E^- = \{A \in \mathrm{CL}(X) : A \cap E \neq \emptyset\}$, $E^+ = \{A \in \mathrm{CL}(X) : A \subset E\}$; further if

$(X, \mathcal{U})$ is a uniform space, put $E^{++} = \{A \in \mathrm{CL}(X) : \exists U \in \mathcal{U} \text{ with } U[A] \subset E\}$, where $U[A] = \{x \in X : \exists a \in A \text{ with } (x, a) \in U\}$. There are three types of hypertopologies which include most of the studied topologies: the *hit-and-miss*, the *proximal hit-and-miss* and the *weak topologies generated by gap and excess functionals* on $\mathrm{CL}(X)$, respectively.

The abstract hit-and-miss topology on $\mathrm{CL}(X)$ has as a subbase all sets of the form V^-, where V is an arbitrary open subset of X plus all sets of the form $(B^c)^+$, where $B^c = X \setminus B$ and B ranges over a fixed nonempty subfamily $\Delta \subset \mathrm{CL}(X)$. It was first studied in [39, 40] and then also in [3, 8, 9, 15, 24, 46, 47, 48]; the well-known prototypes of hit-and-miss topologies are the *Vietoris topology*, with $\Delta = \mathrm{CL}(X)$ ([3, 30, 37]) and the *Fell topology*, with $\Delta =$ nonempty closed compact subsets of X ([3, 19, 30]).

If $(X, \mathcal{U})$ is a uniform space and $(B^c)^+$ is replaced by $(B^c)^{++}$ in the above definition, we get the proximal hit-and-miss topology or hit-and-far topology, studied in [3, 8, 9, 15, 16] and [48], for instance; among its useful prototypes we can find the *proximal topology*, with $\Delta = \mathrm{CL}(X)$ ([5, 17]) or the *ball-proximal topology*, with $\Delta =$ closed proper balls in a metric space X ([11, 25, 48]).

In a metric space (X, d) define the *distance* functional

$$d(x, A) = \inf_{a \in A} d(x, a) \quad (x \in X, \emptyset \neq A \subset X),$$

the *gap* functional

$$D(A, B) = \inf_{a \in A} d(a, B) \quad (A, B \subset X),$$

and the *excess* functional

$$e(A, B) = \sup_{a \in A} d(a, B) \quad (A, B \subset X).$$

Then the so-called *weak hypertopologies* or initial topologies on $\mathrm{CL}(X)$ are defined as the weak topologies generated by gap (in particular distance) and excess functionals, where one of the set arguments of $D(A, B)$ and $e(A, B)$, respectively ranges over given subfamilies of $\mathrm{CL}(X)$. A good reference about these topologies is [3, 4, 26] or [49]. As a prototype of weak hypertopologies we should mention the *Wijsman topology*, which is the weak topology generated by the distance functionals viewed as functionals of set argument ([2, 3, 12, 13, 20, 25, 48, 50]); it is a fundamental tool in the construction of the lattice of hyperspace topologies, as most of the above and many other known topologies arise as suprema and infima, respectively of appropriate Wijsman topologies ([5, 11]). Another weak hypertopology is the *finite Hausdorff topology* which is the weak topology on $\mathrm{CL}(X)$ generated by $\{e(F, \cdot) : F \subset X, F \text{ finite}\} \cup \{e(\cdot, F) : F \subset X, F \text{ finite}\}$ ([3, 26, 50]); the finite Hausdorff topology – as well as the Wijsman topology – is measurably compatible on $\mathrm{CL}(X)$, i.e. its Borel field coincides with the Effros σ-algebra ([3]). Finally, we should at least mention two more important weak hypertopologies, namely the

Hausdorff metric topology and the *Attouch–Wetts topology* ([3]); they however fail to follow the pattern this paper investigates.

In Section 1 of the paper we introduce approach spaces and some other notions needed for the following exposition.

In Section 2 we present a formulation based on the notion of approach spaces, which includes as a special case the abstract hit-and-miss, the proximal hit-and-miss and a big portion of weak hypertopologies (including the Wijsman topology and the finite Hausdorff topology), respectively. One of the advantages of this approach is that it borrows a 'metric-like' quality to the definitions and it may provide a framework where it is easier to come up with results about hypertopologies just by mimicking the results and proofs known for the weak hypertopologies.

In Section 3 we give characterizations of separation axioms T_0, T_1, T_2, T_3 and of complete regularity as well as of metrizability in this general setting using techniques and ideas from [15] and [47].

All this is done to provide a background for proving the main results of the paper in Section 4, where we apply topological games, such as the Banach–Mazur and the strong Choquet game, respectively ([10] or [27]) to investigate properties of hypertopologies; in particular, we characterize Polishness of hypertopologies in this general setting, much in the spirit of [50]. The reason why topological games seem to work well on hyperspaces, and make proofs simpler is that they require the knowledge of basic open sets only, which are the best known and sometimes the only objects to work with, in hypertopologies.

Finally, in Section 5, we 'translate' the general results to get the pertinent applications for (proximal) hit-and-miss and various weak hypertopologies, thus obtaining new results on Polishness of hypertopologies, as well as extending old ones (e.g. [26, 49, 50]).

1. Distances, Gaps and Excesses

Throughout the paper ω stands for the non-negative integers, $\mathscr{P}(X)$ for the power set of X and E^{c} for the complement of $E \subset X$ in X, respectively. If f is a function and E is a set then $f^{\rightarrow}(E)$ (resp. $f^{\leftarrow}(E)$) denotes the image (resp. preimage) of E.

Suppose that (X, δ) is an *approach space* (cf. [31, 32]), i.e. X is a nonempty set and $\delta\colon X \times \mathscr{P}(X) \to [0, \infty]$ is a so-called *distance* (on X) having the following properties:

(D1) $\forall x \in X\colon \delta(x, \{x\}) = 0$,
(D2) $\forall x \in X\colon \delta(x, \emptyset) = \infty$,
(D3) $\forall x \in X\ \forall A, B \subset X\colon \delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}$,
(D4) $\forall x \in X\ \forall A \subset X\ \forall \varepsilon > 0\colon \delta(x, A) \leqslant \delta(x, \mathrm{B}_{\varepsilon}(A)) + \varepsilon$,

where $\mathrm{B}_{\varepsilon}(A) = \{x \in X : \delta(x, A) \leqslant \varepsilon\}$ is the closed ε-hull about A. Every approach space (X, δ) generates a topology τ_{δ} on X defined by the closure operator:

$$\bar{A} = \{x \in X : \delta(x, A) = 0\}, \quad A \subset X.$$

The symbol $S_\varepsilon(A)$ will stand for the 'open' ε-hull about A, which is the set $\{x \in X : \delta(x, A) < \varepsilon\}$ (the reason for the quotation mark is that while the closed hull is closed in (X, τ_δ) as one readily sees from (D4), the 'open' hull is not necessarily open in (X, τ_δ)).

The functional D: $\mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty]$ will be called a *gap* provided:

(G1) $\forall A, B \subset X : D(A, B) \leqslant \inf_{a \in A} \delta(a, B)$,
(G2) $\forall x \in X\ \forall A \subset X : D(\overline{\{x\}}, A) = \inf_{y \in \overline{\{x\}}} \delta(y, A)$,
(G3) $\forall A, B, C \subset X : D(A \cup B, C) = \min\{D(A, C), D(B, C)\}$.

In the sequel (unless otherwise stated) X will stand for an approach space (X, δ) with a gap D (denoted also as (X, δ, D)).

Examples of Distances

- ([31, 32]) Let (X, τ) be a topological space. For $x \in X$ and $A \subset X$ define

$$\delta_t(x, A) = \begin{cases} 0, & \text{if } x \in \bar{A}, \\ \infty, & \text{if } x \notin \bar{A}. \end{cases}$$

  Then δ_t is a distance on X and $\tau = \tau_{\delta_t}$.
- ([33]) Let $(X, \mathcal{U})$ be a uniform space (see [28]). Then $\mathcal{U}$ is generated by the family $\mathcal{D}$ of uniform pseudo-metrics on X such that $d \leqslant 1$ for all $d \in \mathcal{D}$ and

$$d_1, d_2 \in \mathcal{D} \;\Rightarrow\; \max\{d_1, d_2\} \in \mathcal{D}. \tag{1}$$

  Then

$$\delta_u(x, A) = \sup_{d \in \mathcal{D}} d(x, A), \quad x \in X, A \subset X$$

  defines a (bounded) distance on X and τ_{δ_u} coincides with the topology induced by $\mathcal{U}$ on X.
- ([31, 32]) Let (X, d) be a metric space. Then $\delta_m(x, A) = d(x, A)$ is a distance on X and τ_{δ_m} is the topology generated by d on X.

Examples of Gaps

- For an arbitrary approach space (X, δ)

$$D(A, B) = \inf_{a \in A} \delta(a, B)$$

  is clearly a gap on X. This is how we are going to define the gap D_t (resp. D_m) in topological (metric) spaces using the relevant distance δ_t (resp. δ_m).

- Let $(X, \mathcal{U})$ be a uniform space generated by the family $\mathcal{D}$ of uniform pseudometrics on X bounded above by 1. For $A, B \subset X$ define

$$D_u(A, B) = \sup_{d \in \mathcal{D}} \inf_{a \in A} d(a, B).$$

Then D_u is a gap on X. Indeed, it is a routine to check (G1), and (G2) follows from the fact that $D_u(\overline{\{x\}}, A) = \delta_u(x, A)$ for all $x \in X$, $A \subset X$. Finally, (G3) is a consequence of (1) and the boundedness of D_u.

Excess Functional

For all $A, B \subset X$ define the *excess* of A over B by

$$e(A, B) = \sup_{a \in A} \delta(a, B).$$

The symbols e_t, e_u, e_m will stand for the excess in topological, uniform and metric spaces, respectively defined via $\delta_t, \delta_u, \delta_m$. Observe that

(E1) $\forall A, B, C \subset X : e(A \cup B, C) = \max\{e(A, C), e(B, C)\}$,
(E2) $\forall B \subset X\ \forall \varepsilon > 0 : x \in \mathrm{S}_\varepsilon(B) \Rightarrow e(\overline{\{x\}}, B) < \varepsilon$.

Indeed, (E1) is straightforward, and for (E2) notice, that if $e(A, B) < \varepsilon$ for $A, B \subset X$ and $\varepsilon > 0$, then $A \subset \mathrm{B}_\varepsilon(B)$ for some $r > 0$. Hence by closedness of the closed hull about B, we get $\bar{A} \subset \mathrm{B}_{\varepsilon - r}(B)$, so $e(\bar{A}, B) < \varepsilon$.

2. Hyperspace Topologies

In the sequel $\mathrm{CL}(X)$ will stand for the nonempty closed subsets of (X, τ_δ). For $E \subset X$ write $E^- = \{A \in \mathrm{CL}(X) : A \cap E \neq \emptyset\}$, $E^+ = \{A \in \mathrm{CL}(X) : A \subset E\}$ and $E^{++} = \{A \in \mathrm{CL}(X) : D(A, E^c) > 0\}$. Observe that by (G1)

$$E^{++} \subset E^+ \quad \text{for all } E \subset X.$$

In what follows $\Delta_1 \subset \mathrm{CL}(X)$ is arbitrary and $\Delta_2 \subset \mathrm{CL}(X)$ is such that

$$\forall \varepsilon > 0\ \forall A \in \Delta_2 \Rightarrow \mathrm{S}_\varepsilon(A) \text{ is open in } X. \tag{2}$$

As a matter of fact, condition (2) is quite natural and in the applications (see Remark 2.1(iv)) it will be satisfied always.

Denote $\mathfrak{D} = \mathfrak{D}(\Delta_1, \Delta_2) = \bigcup_{k \in \omega} (\Delta_1 \cup \{\emptyset\})^{k+1} \times (\Delta_2 \cup \{X\})^{k+1} \times (0, \infty)^{2k+2}$. Whenever referring to some $S, T \in \mathfrak{D}$ in the sequel, we will assume that for some $k, l \in \omega$

$$S = (S_0, \dots, S_k; \tilde{S}_0, \dots, \tilde{S}_k; \varepsilon_0, \dots, \varepsilon_k; \tilde{\varepsilon}_0, \dots, \tilde{\varepsilon}_k),$$
$$T = (T_0, \dots, T_l; \tilde{T}_0, \dots, \tilde{T}_l; \eta_0, \dots, \eta_l; \tilde{\eta}_0, \dots, \tilde{\eta}_l).$$

For $S \in \mathfrak{D}$ denote

$$M(S) = \bigcap_{i \leqslant k} (\mathrm{B}_{\varepsilon_i}(S_i))^{\mathrm{c}} \cap \mathrm{S}_{\tilde{\varepsilon}_i}(\tilde{S}_i) \quad \text{and}$$

$$S^* = \bigcap_{i \leqslant k} \{A \in \mathrm{CL}(X) : D(A, S_i) > \varepsilon_i \text{ and } e(A, \tilde{S}_i) < \tilde{\varepsilon}_i\}.$$

Observe that if $P = (\emptyset; X; \varepsilon; \tilde{\varepsilon})$, then

$$M(P) = X \quad \text{and} \quad P^* = \mathrm{CL}(X).$$

LEMMA 2.1. *Let $S \in \mathfrak{D}$. Then*

(i) $S^* \subset M(S)^+$;
(ii) $\forall x \in M(S) : ((\exists t > 0\ \forall i \leqslant k\colon \overline{\{x\}} \cap \mathrm{S}_{\varepsilon_i + t}(S_i) = \emptyset) \Rightarrow \overline{\{x\}} \in S^*)$;
(iii) $A \subset B \in S^* \Rightarrow A \in S^*$.

Proof. (i) If $A \in S^*$ then by (G1), $\varepsilon_i < D(A, S_i) \leqslant \inf_{x \in A} \delta(x, S_i)$, so $A \subset (\mathrm{B}_{\varepsilon_i}(S_i))^{\mathrm{c}}$ for all $i \leqslant k$. Further $\tilde{\varepsilon}_i > e(A, \tilde{S}_i) = \sup_{x \in A} \delta(x, \tilde{S}_i)$ implies that $A \subset \mathrm{S}_{\tilde{\varepsilon}_i}(\tilde{S}_i)$.

(ii) Use (G2) and (E2).

(iii) Use that by (G3) (resp. (E1)) $D(A, C) \geqslant D(B, C)$ (resp. $e(A, C) \leqslant e(B, C)$) for all $C \subset X$. □

For $U_0, \ldots, U_n \in \tau_\delta$ and $S \in \mathfrak{D}$ denote

$$(U_0, \ldots, U_n)_S = \bigcap_{i \leqslant n} U_i^- \cap S^*.$$

It is easy to see that the collection

$$\mathcal{B}^* = \{(U_0, \ldots, U_n)_S : U_0, \ldots, U_n \in \tau_\delta, S \in \mathfrak{D}, n \in \omega\}$$

forms a base for a topology on $\mathrm{CL}(X)$; denote it by τ^*.

Remark 2.1. (i) Let (X, τ) be a topological space. Let $\Delta_1 = \Delta$ and $\Delta_2 = \{X\}$. Then for $B \in \Delta$ and $\varepsilon, \eta > 0$, $\{A \in \mathrm{CL}(X) : D_t(A, B) > \varepsilon\} = (B^{\mathrm{c}})^+$ and $\{A \in \mathrm{CL}(X) : e_t(A, X) < \eta\} = \mathrm{CL}(X)$. Thus $\tau^* = \tau^+$ is the general *hit-and-miss topology* on $\mathrm{CL}(X)$.

(ii) Let $(X, \mathcal{U})$ be a uniform space. Let $\Delta_1 = \Delta$ and $\Delta_2 = \{X\}$. Then for $B \in \Delta$ and $\varepsilon, \eta > 0$, $\{A \in \mathrm{CL}(X) : D_u(A, B) > \varepsilon\} = (B^{\mathrm{c}})^{++}$ and $\{A \in \mathrm{CL}(X) : e_u(A, X) < \eta\} = \mathrm{CL}(X)$. Thus $\tau^* = \tau^{++}$ is the *proximal hit-and-miss topology* on $\mathrm{CL}(X)$.

(iii) Let (X, d) be a metric space. Let $\Delta_1, \Delta_2 \subset \mathrm{CL}(X)$ be such that Δ_1 contains the singletons. Then τ^* coincides with the weak hypertopology τ_{weak} generated by gap and excess functionals.

(iv) Observe that in the above applications, condition (2) is not really restrictive, since in uniform and metric spaces it is fulfilled for arbitrary $\Delta_2 \subset \mathrm{CL}(X)$; further for the hit-and-miss topology we need only $\Delta_2 = \{X\}$ thus, in this setting (2) holds again.

3. Separation Axioms and Metrizability

We will say that $\mathfrak{D}$ is a *Urysohn family* provided whenever $S \in \mathfrak{D}$ and $A \in S^*$ there exists $T \in \mathfrak{D}$ such that $A \in T^*$ and $\overline{M(T)} \in S^*$.

Further $\mathfrak{D}$ is a *weakly Urysohn family* provided for all $S \in \mathfrak{D}$ and $A \in S^*$ there exists $T \in \mathfrak{D}$ such that $A \in T^* \subset S^*$ and

$$\forall E \text{ countable: } E \subset M(T) \Longrightarrow \overline{E} \in S^*. \tag{3}$$

The family $\mathfrak{D}$ is *weakly quasi-Urysohn* provided for all $\emptyset \neq (U_0, \dots, U_n)_S \in \mathcal{B}^*$ there is a $T \in \mathfrak{D}$ such that $\emptyset \neq (U_0, \dots, U_n)_T \subset (U_0, \dots, U_n)_S$ and S, T satisfies (3).

Remark 3.1. (i) If $\mathfrak{D}$ is a Urysohn family, it is also weakly Urysohn (by Lemma 2.1(iii)); if $\mathfrak{D}$ is a weakly Urysohn family, it is a weakly quasi-Urysohn family as well.

(ii) Let (X, τ) ($(X, \mathcal{U})$) be a topological (uniform) space, $\Delta_2 = \{X\}$ and $\Delta_1 = \Delta$. Denote by $\Sigma(\Delta)$ the finite unions of members of $\Delta \subset \mathrm{CL}(X)$.

If Δ is a (*uniformly*) *Urysohn family* (i.e. for all $S \in \Sigma(\Delta)$ and $A \in (S^c)^+$ (resp. $A \in (S^c)^{++}$) there exists a $T \in \Sigma(\Delta)$ with $A \subset T^c \subset \overline{T^c} \subset S^c$ (resp. $A \subset T^c \subset U[S]^c$ for some $U \in \mathcal{U}$) – cf. [2, 3]), then $\mathfrak{D}_t (= \mathfrak{D}_u) = \mathfrak{D}(\Delta, \{X\})$ is a Urysohn family.

Further, if we will say that Δ is a (*uniformly*) *weakly Urysohn family* provided for all $S \in \Sigma(\Delta)$ and $A \in (S^c)^+$ (resp. $A \in (S^c)^{++}$) there exists a $T \in \Sigma(\Delta)$ with $A \subset T^c \subset S^c$ (resp. $A \subset T^c \subset U[S]^c$ for some $U \in \mathcal{U}$) such that

$$\forall E \text{ countable: } (E \subset T^c \Longrightarrow \overline{E} \subset S^c) \quad (\text{resp. } \overline{E} \subset U[S]^c \text{ for some } U \in \mathcal{U}), \tag{3'}$$

then $\mathfrak{D}_t (= \mathfrak{D}_u)$ is a weakly Urysohn family.

Similarly, if Δ is said to be a (*uniformly*) *weakly quasi-Urysohn family* provided whenever $S \in \Sigma(\Delta)$ and $U_i \in \tau \setminus \{\emptyset\}$ are disjoint for all $i \leqslant n$, there exists $T \in \Sigma(\Delta)$ such that $U_i \cap T^c \neq \emptyset$ for all $i \leqslant n$, $S \subset T$ and (3′) holds, then $\mathfrak{D}_t (= \mathfrak{D}_u)$ is a weakly quasi-Urysohn family.

Note that a (*uniformly*) *quasi-Urysohn family* of [48] is a (uniformly) weakly quasi-Urysohn family.

(iii) If (X, d) is a metric space and Δ_1 contains the singletons, then $\mathfrak{D}_m = \mathfrak{D}(\Delta_1, \Delta_2)$ is a Urysohn (hence also weakly Urysohn and weakly quasi-Urysohn) family.

Indeed, suppose that $S \in \mathfrak{D}_m$. Then $A \in S^*$ implies $D_m(A, S_i) > \varepsilon_i$ and $e_m(A, \tilde{S}_i) < \tilde{\varepsilon}_i$ for all $i \leqslant k$. Put $\eta_i = \frac{\varepsilon_i + D_m(A,S_i)}{2}$ and $\tilde{\eta}_i = \frac{\tilde{\varepsilon}_i + e_m(A,\tilde{S}_i)}{2}$ for $i \leqslant k$ and define $T = (S_0, \dots, S_k; \tilde{S}_0, \dots, \tilde{S}_k; \eta_0, \dots, \eta_k; \tilde{\eta}_0, \dots, \tilde{\eta}_k)$. Then clearly $A \in T^*$. Further, if $x \in \overline{M(T)}$ then $d(x, S_i) \geqslant \eta_i$ and $d(x, \tilde{S}_i) \leqslant \tilde{\eta}_i$ for all $i \leqslant k$, so $D_m(\overline{M(T)}, S_i) \geqslant \eta_i > \varepsilon_i$ and $e_m(\overline{M(T)}, \tilde{S}_i) \leqslant \tilde{\eta}_i < \tilde{\varepsilon}_i$ for all $i \leqslant k$. It means that $\overline{M(T)} \in S^*$.

We will say that (X, δ, D) has *property* (P) provided

$$\forall x \in X\ \forall k \in \omega\ \forall A_0, \dots, A_k \in \mathrm{CL}(X)\ \forall \varepsilon_0, \dots, \varepsilon_k > 0\ \exists y \in \overline{\{x\}}:$$
$$\delta(x, A_i) > \varepsilon_i \Rightarrow D(\overline{\{y\}}, A_i) > \varepsilon_i \quad \text{for all } i \leqslant k.$$

Remark 3.2. (i) Observe that (P) is satisfied in uniform and metric spaces, since there $\delta(x, A) = D(\overline{\{x\}}, A)$.

(ii) In a topological space (X, τ), (P) is satisfied iff X is *weakly*-R_0 (see [46, 47]), i.e. for all $U \in \tau$ and $x \in U$ there is a $y \in \overline{\{x\}}$ with $\overline{\{y\}} \subset U$ iff each nonempty difference of open sets contains a nonempty closed set.

The following lemma will play a key role in the next paragraph as it will enable the transfer of topological games from the base space to the hyperspace.

LEMMA 3.1. *Suppose that* (X, δ, D) *has property* (P). *Suppose that* $U_0, \dots, U_n$, $V_0, \dots, V_m \in \tau_\delta$ *for some* $m, n \in \omega$ *and* $S, T \in \mathfrak{D}$.

Then $\emptyset \neq (U_0, \dots, U_n)_S \subset (V_0, \dots, V_m)_T$ *implies* $M(S) \subset M(T)$ *and for all* $j \leqslant m$ *there exists* $i \leqslant n$ *with* $M(S) \cap U_i \subset M(T) \cap V_j$.

Proof. Let $A \in \mathbf{U} = (U_0, \dots, U_n)_S$ and $\mathbf{V} = (V_0, \dots, V_m)_T$. Suppose that there exists an $x \in M(S) \setminus M(T)$. Then by (P), there exists $y \in \overline{\{x\}}$ such that $D(\overline{\{y\}}, S_r) > \varepsilon_r$ for all $r \leqslant k$. Then $\overline{\{y\}} \in S^*$, since by (E2) and $y \in \overline{\{x\}}$ we have $e(\overline{\{y\}}, \tilde{S}_r) \leqslant e(\overline{\{x\}}, \tilde{S}_r) < \tilde{\varepsilon}_r$ for all $r \leqslant k$.

Further $y \notin M(T)$. Indeed, since $x \notin M(T)$ then either $x \in \mathrm{B}_{\eta_s}(T_s)$ or $x \notin \mathrm{S}_{\tilde{\eta}_s}(\tilde{T}_s)$ for some $s \leqslant l$. In the first case using closedness of the closed hull we get that $y \in \overline{\{x\}} \subset \mathrm{B}_{\tilde{\eta}_s}(T_s)$, while in the second case in view of (2), $y \in \overline{\{x\}} \subset (\mathrm{S}_{\tilde{\eta}_s}(\tilde{T}_s))^{\mathrm{c}}$. Now by (G3) and (E1), $A \cup \overline{\{y\}} \in \mathbf{U}$, but by Lemma 2.1(i), $A \cup \overline{\{y\}} \notin \mathbf{V}$ since $A \cup \overline{\{y\}} \notin M(T)^+$, which is a contradiction.

Suppose that there exists $j \leqslant m$ such that for all $i \leqslant n$ we have some $x_i \in U_i \cap M(S) \setminus V_j \cap M(T)$. Then for all $i \leqslant n$, $\delta(x_i, U_i^{\mathrm{c}}) > \theta_i$ for some $\theta_i > 0$ and $x_i \notin V_j$ or $x_i \notin M(T)$. Then similarly to the above reasoning for all $i \leqslant n$ we can find a $y_i \in \overline{\{x_i\}}$ such that $\overline{\{y_i\}} \in U_i^- \cap S^*$ and $y_i \notin V_j \cap M(T)$. Hence by (G2) and (E1), $\bigcup_{i \leqslant n} \overline{\{y_i\}} \in \mathbf{U}$, but in view of Lemma 2.1(i), $\bigcup_{i \leqslant n} \overline{\{y_i\}} \notin \mathbf{V}$, and we are done. □

It is easy to show that:

THEOREM 3.1. $(\mathrm{CL}(X), \tau^*)$ *is* T_0.

The following theorem generalizes results from [39] (Satz 1) and [23] (Theorem 3.11):

THEOREM 3.2. *The following are equivalent:*

(i) $(\mathrm{CL}(X), \tau^*)$ *is* T_1*;*
(ii) $\forall A \in \mathrm{CL}(X)\ \forall x \in A^c\ \exists S \in \mathfrak{D} : A \in S^*$ *and* $M(S)^c \cap \overline{\{x\}} \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii) Since $A \neq A \cup \overline{\{x\}}$, by T_1-ness of the hyperspace we can find some $\mathbf{U} = (U_0, \ldots, U_n)_S \in \mathcal{B}^*$ containing A and not containing $A \cup \overline{\{x\}}$. Then $A \in S^*$ and if $x \notin M(S)$, we are done. Thus, assume that $x \in M(S)$ and let

$$t = \frac{1}{2} \min_{i \leqslant k} \{D(A, S_i) - \varepsilon_i, \delta(x, S_i) - \varepsilon_i\}.$$

Define $\hat{S} = (S_0, \ldots, S_k; \tilde{S}_0, \ldots, \tilde{S}_k; \varepsilon_0 + t, \ldots, \varepsilon_k + t; \tilde{\varepsilon}_0, \ldots, \tilde{\varepsilon}_k)$. Then $A \in \hat{S}^*$ and $M(\hat{S})^c \cap \overline{\{x\}} \neq \emptyset$. Indeed, if $\overline{\{x\}} \subset M(\hat{S})$, then by Lemma 2.1(ii), $\overline{\{x\}} \in S^*$. Hence by (G3) and (E1) we would have that $A \cup \overline{\{x\}} \in \mathbf{U}$, which is a contradiction.

(ii) $\Rightarrow$ (i) Pick $A, B \in \mathrm{CL}(X)$, $A \neq B$ with (say) $x \in B \setminus A$. If $S \in \mathfrak{D}$ is taken in accordance with (ii), then $A \in S^*$ and in view of Lemma 2.1(i), $B \notin S^*$ since $\overline{\{x\}} \not\subseteq M(S)$. Further B is separated from A by $(A^c)^-$. □

COROLLARY 3.1. *If* $(\mathrm{CL}(X), \tau^*)$ *is* T_1 *then*

(i) X *has property* (P);
(ii) X *is weakly-*R_0.

Proof. (i) Take some $A_0, \ldots, A_k \in \mathrm{CL}(X)$ and some $\varepsilon_0, \ldots, \varepsilon_k > 0$, where $k \in \omega$. Pick an $x \in X$ such that $\delta(x, A_i) > \varepsilon_i$ for all $i \leqslant k$. Let

$$t = \frac{1}{2} \min_{i \leqslant k} \{\delta(x, A_i) - \varepsilon_i\}.$$

Then it is easy to see that $x \notin A = \bigcup_{i \leqslant k} \mathrm{B}_{\varepsilon_i + t}(A_i)$. Now clearly $A \in \mathrm{CL}(X)$, so by Theorem 3.2(ii) we can find an $S \in \mathfrak{D}$ such that

$$A \in S^* \quad \text{and} \quad M(S)^c \cap \overline{\{x\}} \neq \emptyset.$$

Therefore we can take some $y \in M(S)^c \cap \overline{\{x\}}$. Observe that $M(S)^c$ is closed in X, so $\overline{\{y\}} \subset M(S)^c$, further by Lemma 2.1(i), $A \in S^* \subset M(S)^+$, which means that $\overline{\{y\}} \subset M(S)^c \subset A^c$. It follows then that $\delta(z, A_i) > \varepsilon_i + t$ for all $z \in \overline{\{y\}}$ and $i \leqslant k$. Consequently, by (G2) we have that $D(\overline{\{y\}}, A_i) = \inf_{z \in \overline{\{y\}}} \delta(z, A_i) \geqslant \varepsilon_i + t > \varepsilon_i$ for every $i \leqslant k$, which justifies property (P) for X.

(ii) It follows from the fact that X is weakly-R_0 whenever X has property (P). Indeed, let U be open in (X, τ_δ) and $x \in U$. Then $\delta(x, U^c) > \varepsilon$ for some $\varepsilon > 0$ and hence by property (P), there is a $y \in \overline{\{x\}}$ with $D(\overline{\{y\}}, U^c) > \varepsilon$. So by (G2), $\delta(z, U^c) > \varepsilon$ for all $z \in \overline{\{y\}}$, whence $\overline{\{y\}} \subset U$. □

We will denote by $\mathrm{cl}_*(\mathbf{E})$ the closure of $\mathbf{E} \subset \mathrm{CL}(X)$ in $(\mathrm{CL}(X), \tau^*)$.

LEMMA 3.2. *Let (X, τ_δ) be a weakly-R_0 space. Let $\mathbf{U} = (U_0, \dots, U_n)_S \in \mathcal{B}^*$ and $A_0 \in \mathbf{U}$. Then $A \in \mathrm{cl}_*(\mathbf{U})$ for all $A \in \mathrm{CL}(X)$ with $A_0 \subset A \subset \overline{M(S)}$.*

Proof. Let $\mathbf{V} = (V_0, \dots, V_m)_T \in \mathcal{B}^*$ be a neighborhood of A. For all $j \leqslant m$ let $a_j \in A \cap V_j$. Then $a_j \in M(T)$ by Lemma 2.1(i); thus, $V_j \cap M(T)$ is an open neighborhood of $a_j \in A$ and hence there exists $x_j \in V_j \cap M(T) \cap M(S)$. For every $j \leqslant m$, put

$$t_j = \frac{1}{2} \min \left\{ \min_{i \leqslant k}\{\delta(x_j, S_i) - \varepsilon_i\}, \min_{h \leqslant l}\{\delta(x_j, T_h) - \eta_h\} \right\}.$$

Then $x_j \in V_j \cap \bigcap_{i \leqslant k}(\mathrm{B}_{\varepsilon_i + t_j}(S_i))^{\mathrm{c}} \cap \bigcap_{h \leqslant l}(\mathrm{B}_{\eta_h + t_j}(T_h))^{\mathrm{c}}$ and by weak-R_0-ness of X there exist $y_j \in \overline{\{x_j\}}$ with $\overline{\{y_j\}} \subset V_j$, $\overline{\{y_j\}} \cap \mathrm{B}_{\varepsilon_i + t_j}(S_i) = \emptyset$ for $i \leqslant k$ and $\overline{\{y_j\}} \cap \mathrm{B}_{\eta_h + t_j}(T_h) = \emptyset$ for $h \leqslant l$. The last two relations imply by Lemma 2.1(ii) that $\overline{\{y_j\}} \in S^* \cap T^*$. Since $A_0 \in S^*$ and, by Lemma 2.1(iii), $A_0 \in T^*$ (because $A_0 \subset A \in T^*$), we have by (G3) and (E1) that $A_0 \cup \hat{A} \in S^* \cap T^*$, where $\hat{A} = \bigcup_{j \leqslant m} \overline{\{y_j\}}$. Also, $(A_0 \cup \hat{A}) \cap V_j \supset \overline{\{y_j\}} \neq \emptyset$ for $j \leqslant m$, and $(A_0 \cup \hat{A}) \cap U_i \supset A_0 \cap U_i \neq \emptyset$ for $i \leqslant n$. All this means that $A_0 \cup \hat{A} \in \mathbf{V} \cap \mathbf{U}$, so $A \in \mathrm{cl}_*(\mathbf{U})$. □

Concerning the following theorem see [39] (Satz 3) and [46] (Theorem 2):

THEOREM 3.3. *The following are equivalent:*

(i) *$(\mathrm{CL}(X), \tau^*)$ is T_2;*
(ii) *$\forall A \in \mathrm{CL}(X)\ \forall x \in A^{\mathrm{c}}\ \exists S \in \mathfrak{D} : A \in S^*$ and $x \in \overline{M(S)}^{\mathrm{c}}$.*

Proof. (i) ⇒ (ii) Let $\mathbf{U} = (U_0, \dots, U_n)_S \in \mathcal{B}^*$ be a neighborhood of A such that $\mathrm{cl}_*(\mathbf{U})$ does not contain $A \cup \overline{\{x\}}$. Then $A \in S^*$. If $x \in \overline{M(S)}$, then $\overline{\{x\}} \subset \overline{M(S)}$ and hence by Corollary 3.1(ii) and Lemma 3.2, $A \cup \overline{\{x\}} \in \mathrm{cl}_*(\mathbf{U})$, which is a contradiction.

(ii) ⇒ (i) Let $A, B \in \mathrm{CL}(X)$ be distinct and (say) $x \in B \setminus A$. Choose an $S \in \mathfrak{D}$ as in (ii). Then by Lemma 2.1(i), S^* and $(\overline{M(S)}^{\mathrm{c}})^-$ are disjoint τ^*-neighborhoods of A and B, respectively. □

Denote by $\mathcal{F}_{\mathfrak{D}}$ the class of all continuous functions $f \colon X \to [0, 1]$ such that whenever $\inf f < a < b < \sup f$ there exists $S \in \mathfrak{D}$ with

$$f^{\leftarrow}([b, 1]) \in S^* \quad \text{and} \quad \overline{M(S)} \subset f^{\leftarrow}([a, 1]).$$

LEMMA 3.3. *The infimal value functional $m_f(A) = \inf_{x \in A} f(x)$ on $\mathrm{CL}(X)$ is τ^*-continuous for all $f \in \mathcal{F}_{\mathfrak{D}}$.*

Proof. Let $\inf f < a < b < \sup f$ and $A \in m_f^{\leftarrow}((a, b))$. Then $A \cap f^{\leftarrow}((a, b)) \neq \emptyset$ and for any $0 < \varepsilon < m_f(A) - a$ we have $A \subset f^{\leftarrow}([a + \varepsilon, 1])$. Now take an

$S \in \mathfrak{D}$ such that $f^{\leftarrow}([a+\varepsilon,1]) \in S^*$ and $\overline{M(S)} \subset f^{\leftarrow}([a+\varepsilon/2,1])$. Then by Lemma 2.1(iii) and (i), $A \in S^* \cap (f^{\leftarrow}((a,b)))^- \subset m_f^{\leftarrow}((a,b))$.

The cases $m_f(A) = \inf f$ and $m_f(A) = \sup f$, respectively are easier. □

Confer Lemma 3.1 of [9] (or Lemma 4.4.3 and 4.4.7 of [3]) with the following:

LEMMA 3.4. *The following are equivalent:*

(i) $\mathfrak{D}$ *is a Urysohn family;*
(ii) *for all* $S \in \mathfrak{D}$ *and* $A \in S^*$ *there exists* $f \in \mathcal{F}_{\mathfrak{D}}$ *such that* $f^{\rightarrow}(A) = \{1\}$ *and* $m_f^{\rightarrow}((S^*)^{\mathrm{c}}) = \{0\}$.

Proof. (i) ⇒ (ii) Put $S_0 = S$. Proceed inductively, using that $\mathfrak{D}$ is a Urysohn family, to get a collection $S_{n/2^k} \in \mathfrak{D}$ ($n, k \in \omega$ and $0 \leqslant n/2^k < 1$) such that

$$A \in S^*_{(n+1)/2^k} \quad \text{and} \quad \overline{M(S_{(n+1)/2^k})} \in S^*_{n/2^k} \quad (k \in \omega, 0 \leqslant n < 2^k - 1). \tag{4}$$

Indeed, suppose (4) is fulfilled for $k \leqslant \hat{k}$ and $0 \leqslant n < 2^k - 1$, and let $\hat{n}$ be such that $0 \leqslant \hat{n} < 2^{\hat{k}} - 1$. Then from $\overline{M(S_{(\hat{n}+1)/2^{\hat{k}}})} \in S^*_{\hat{n}/2^{\hat{k}}}$, we get an $S_{(2\hat{n}+1)/2^{\hat{k}+1}} \in \mathfrak{D}$ such that $\overline{M(S_{(\hat{n}+1)/2^{\hat{k}}})} \in S^*_{(2\hat{n}+1)/2^{\hat{k}+1}}$ and $\overline{M(S_{(2\hat{n}+1)/2^{\hat{k}+1}})} \in S^*_{\hat{n}/2^{\hat{k}}}$; thus, (4) holds both for $n = 2\hat{n}$ and $k = \hat{k}+1$. Moreover, $A \in S^*_{(2\hat{n}+1)/2^{\hat{k}+1}}$ is a consequence of $A \in S^*_{(\hat{n}+1)/2^{\hat{k}}}$, $\overline{M(S_{(2\hat{n}+1)/2^{\hat{k}+1}})} \in S^*_{\hat{n}/2^{\hat{k}}}$, and Lemma 2.1(i) and (ii).

It is clear by Lemma 2.1(i) that

$$\overline{M(S_{(n+1)/2^k})} \subset M(S_{n/2^k}). \tag{5}$$

Define $f\colon X \to [0,1]$ via

$$f(x) = \begin{cases} \inf\{r \in [0,1) : r \text{ a dyadic rational and } x \in \overline{M(S_r)}^{\mathrm{c}}\}, & \text{if } x \in \bigcup_r \overline{M(S_r)}^{\mathrm{c}}, \\ 1, & \text{otherwise.} \end{cases}$$

Then $f^{\rightarrow}(A) = \{1\}$. Now assume that $E \notin S^*$ and $E \subset \overline{M(S_{(n+1)/2^k})}$ for some $k > 0$ and $0 \leqslant n \leqslant 2^k - 2$. In view of (5), $E \subset \overline{M(S_{1/2^k})}$ so by (4) and Lemma 2.1(iii), $E \in S_0^*$, which is a contradiction. It means that if $E \notin S^*$, then for all dyadic rationals $r \in (0,1)$, $E \cap (\overline{M(S_r)})^{\mathrm{c}} \neq \emptyset$. Consequently, $m_f(E) = \inf_{x\in E} f(x) \leqslant r$ for all dyadic $r \in (0,1)$, whence $m_f(E) = 0$.

To justify that $f \in \mathcal{F}_{\mathfrak{D}}$, observe that if $0 < a < b < 1$ and $n, k \in \omega$ is such that $a < n/2^k < (n+1)/2^k < b$, then $f^{\leftarrow}([b,1]) \subset \overline{M(S_{(n+1)/2^k})} \in S^*_{n/2^k}$, whence $f^{\leftarrow}([b,1]) \in S^*_{n/2^k}$ by Lemma 2.1(iii); on the other hand, it easily follows from (5) that $\overline{M(S_{n/2^k})} \subset f^{\leftarrow}([a,1])$. Finally, notice that for any $a \in (0,1)$

$$f^{\leftarrow}([0,a)) = \bigcup_{\substack{r\in[0,a)\\ r \text{ dyadic}}} \overline{M(S_r)}^{\mathrm{c}} \text{ and by (5),}$$

$$f^{\leftarrow}((a,1]) = \bigcup_{\substack{r\in(a,1)\\ r \text{ dyadic}}} M(S_r)$$

are open in X, so f is continuous.

(ii) $\Rightarrow$ (i) Since $f \in \mathcal{F}_{\mathfrak{D}}$, we can find a $T \in \mathfrak{D}$ with $f^{\leftarrow}([1/2, 1]) \in T^*$ and $\overline{M(T)} \subset f^{\leftarrow}([1/4, 1])$. Then by Lemma 2.1(iii) $A \in T^*$, since $A \subset f^{\leftarrow}([1/2, 1])$, further $\overline{M(T)} \in S^*$, since $m_f(\overline{M(T)}) \geqslant \frac{1}{4} > 0$. □

The origins of the following theorem are in [40] (Satz 6), in [9] (Theorems 3.3 and 3.6) and in [47] (Theorem 2.8):

THEOREM 3.4. *The following are equivalent:*

(i) $(\mathrm{CL}(X), \tau^*)$ *is Tychonoff;*
(ii) $(\mathrm{CL}(X), \tau^*)$ *is completely regular;*
(iii) $(\mathrm{CL}(X), \tau^*)$ *is regular;*
(iv) $(\mathrm{CL}(X), \tau^*)$ *is* T_3*;*
(v) $(\mathrm{CL}(X), \tau^*)$ *is* T_1 *and* $\mathfrak{D}$ *is a Urysohn family.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (iv) By Theorem 3.1, the hyperspace is always T_0 and hence T_1.

(iv) $\Rightarrow$ (v) Let $\mathbf{V} = (V_0, \ldots, V_n)_T \in \mathcal{B}^*$ be such that $A \in \mathbf{V} \subset \mathrm{cl}_*(\mathbf{V}) \subset S^*$. Then $A \in T^*$ and by Corollary 3.1(ii) and Lemma 3.2, $\overline{M(T)} \in \mathrm{cl}_*(\mathbf{V}) \subset S^*$.

(v) $\Rightarrow$ (i) Let $A \in \mathrm{CL}(X)$ and $\mathbf{U} = (U_0, \ldots, U_n)_S \in \mathcal{B}^*$ be a neighborhood of A. Without loss of generality assume that $U_i \subsetneq X$ for all $i \leqslant n$. Then $A \in S^*$ and $A \cap U_i \neq \emptyset$ for all $i \leqslant n$. It follows from Theorem 3.2 that for each $i \leqslant n$ we can find $T_i \in \mathfrak{D}$ such that $U_i^{\mathrm{c}} \in T_i^*$ and $M(T_i)^{\mathrm{c}} \cap A \neq \emptyset$. By virtue of Lemma 3.4 we get some $f_0, \ldots, f_n, g \in \mathcal{F}_{\mathfrak{D}}$ such that

$$g^{\rightarrow}(A) = \{1\} \quad \text{and} \quad m_g^{\rightarrow}((S^*)^{\mathrm{c}}) = \{0\}, \tag{6}$$

$$f_i^{\rightarrow}(U_i^{\mathrm{c}}) = \{1\} \quad \text{and} \quad m_{f_i}^{\rightarrow}((T_i^*)^{\mathrm{c}}) = \{0\} \quad \text{for each} \quad i \leqslant n. \tag{7}$$

Then by Lemma 3.3, $m_{f_0}, \ldots, m_{f_n}, m_g$ are τ^*-continuous on $\mathrm{CL}(X)$ and so is $\varphi = \max\{1-m_g, m_{f_0}, \ldots, m_{f_n}\}$. Clearly, $1-m_g(A) = m_{f_0}(A) = \cdots = m_{f_n}(A) = 0$, whence $\varphi(A) = 0$. Further, if $E \notin \mathbf{U}$ then either $E \notin S^*$ or $E \cap U_i = \emptyset$ for some $i \leqslant n$. In the first case, $m_g(E) = 0$ by (6), while in the second case $m_{f_i}(E) = 1$ by (7). Consequently $\varphi(E) = 1$ for all $E \in \mathbf{U}^{\mathrm{c}}$ and complete regularity of $(\mathrm{CL}(X), \tau^*)$ follows. □

COROLLARY 3.2. *If* $(\mathrm{CL}(X), \tau^*)$ *is regular or* T_2*, then* (X, τ_δ) *is regular.*

Proof. In view of Theorem 3.4, $(\mathrm{CL}(X), \tau^*)$ is T_3 and hence T_2. Thus, if $A \in \mathrm{CL}(X)$ and $x \in A^{\mathrm{c}}$ then by Theorem 3.3 there exists $S \in \mathfrak{D}$ with $A \in S^*$ and $x \in \overline{M(S)}^{\mathrm{c}}$. Finally observe that by Lemma 2.1(i), $\overline{\overline{M(S)}^{\mathrm{c}}} \subset M(S)^{\mathrm{c}} \subset A^{\mathrm{c}}$, which justifies regularity of X. □

The following result generalizes Theorems 5.19 and 5.20 of [15] and Theorem 2.10 of [47] (see also Theorem 3.2 and Proposition 4.1 in [26]):

THEOREM 3.5. *The following are equivalent:*

(i) $(\mathrm{CL}(X), \tau^*)$ *is metrizable;*
(ii) $(\mathrm{CL}(X), \tau^*)$ *is pseudo-metrizable;*
(iii) $(\mathrm{CL}(X), \tau^*)$ *is 2nd countable and regular;*
(iv) $(\mathrm{CL}(X), \tau^*)$ *is* T_1 *and* $\mathfrak{D}$ *contains a countable Urysohn subfamily (i.e.* $\exists$ *countable* $\mathfrak{D}' \subset \mathfrak{D}\ \forall S \in \mathfrak{D}\ \forall A \in S^*\ \exists T \in \mathfrak{D}' : A \in T^*$ *and* $\overline{M(T)} \in S^*$*).*

Proof. (i) $\Leftrightarrow$ (ii) Follows from Theorem 3.4.

(i) $\Rightarrow$ (iii) Use Corollary 3.1(i) and Lemma 3.1 to justify that 1st countability of $(\mathrm{CL}(X), \tau^*)$ implies separability of X. Let E be a countable dense subset of X. Then the collection $\mathbf{E}$ of the closures of finite subsets of E is countable and dense in $(\mathrm{CL}(X), \tau^*)$. To see density take some nonempty $\mathbf{U} = (U_0, \ldots, U_n)_S \in \mathcal{B}^*$. Since $\mathfrak{D}$ is a Urysohn family (cf. Theorem 3.4), we can find $T \in \mathfrak{D}$ with $\overline{M(T)} \in S^*$ and $U_i \cap M(T) \neq \emptyset$ for all $i \leqslant n$. Pick some $x_i \in E \cap U_i \cap M(T)$ for each $i \leqslant n$. Then by Lemma 2.1(iii), $F = \bigcup_{i \leqslant n} \overline{\{x_i\}} \in S^*$, so $F \in \mathbf{E} \cap \mathbf{U}$. Thus $(\mathrm{CL}(X), \tau^*)$ is a separable metric space and hence 2nd countable.

(iii) $\Rightarrow$ (iv) Regularity of $(\mathrm{CL}(X), \tau^*)$ implies its T_1-ness (see the proof of Theorem 3.4 (iii) $\Rightarrow$ (iv)). Further, if $\mathcal{B}' \subset \mathcal{B}^*$ is a countable base of $(\mathrm{CL}(X), \tau^*)$ (cf. [18], Theorem 1.1.15), define $\mathfrak{D}'$ as the subcollection of $\mathfrak{D}$ consisting of the elements appearing in the representation of members of $\mathcal{B}'$. Using that $\mathfrak{D}$ is a Urysohn family, one can easily justify that $\mathfrak{D}'$ satisfies (iv).

(iv) $\Rightarrow$ (i) It is clear by Theorem 3.4 that the hyperspace is T_3. Further, if X had a countable base $\mathcal{B}$, then $\{(U_0, \ldots, U_n)_S \in \mathcal{B}^* : U_0, \ldots, U_n \in \mathcal{B}, S \in \mathfrak{D}', n \in \omega\}$ would be a countable base of $(\mathrm{CL}(X), \tau^*)$ and the Urysohn Metrization Theorem would yield (i). Hence it remains to show that $\mathcal{B} = \{\overline{M(T)}^{\mathrm{c}} : T \in \mathfrak{D}'\} \cup \{X\}$ is a countable base for (X, τ_δ). Indeed, if $x \in U \in \tau_\delta$ and (without loss of generality) $U \neq X$, then by Theorem 3.2 we get an $S \in \mathfrak{D}$ with $U^{\mathrm{c}} \in S^*$ and $\overline{\{x\}} \cap M(S)^{\mathrm{c}} \neq \emptyset$. Pick a $T \in \mathfrak{D}'$ such that $U^{\mathrm{c}} \in T^*$ and $\overline{M(T)} \in S^*$ (from (iv)). Then $\overline{\{x\}} \cap \overline{M(T)}^{\mathrm{c}} \supset \overline{\{x\}} \cap M(S)^{\mathrm{c}} \neq \emptyset$, so $x \in \overline{M(T)}^{\mathrm{c}}$. Finally, $U^{\mathrm{c}} \in T^*$ implies by Lemma 2.1(i) that $U^{\mathrm{c}} \subset M(T) \subset \overline{M(T)}$, whence $\overline{M(T)}^{\mathrm{c}} \subset U$. $\square$

4. Topological Games and Hyperspace Topologies

Let $(Y, \mathcal{B})$ be a topological space with an open base $\mathcal{B}$. The *strong Choquet game* on Y is played as follows: two players α and B$_\varepsilon ta$ take turns in choosing objects in $(Y, \mathcal{B})$; β starts by picking (x_0, V_0) from

$$\mathcal{E}(Y, \mathcal{B}) = \{(x, V) \in Y \times \mathcal{B} : x \in V\}$$

and α responds by some $U_0 \in \mathcal{B}$ with $x_0 \in U_0 \subset V_0$. The next choice of β is a couple $(x_1, V_1) \in \mathcal{E}(Y, \mathcal{B})$ such that $x_1 \in V_1 \subset U_0$ and then α picks a $U_1 \in \mathcal{B}$ with $x_1 \in U_1 \subset V_1$ etc. Player α wins the run $(x_0, V_0), U_0, \ldots, (x_p, V_p), U_p, \ldots$ provided

$$\bigcap_{p \in \omega} V_p = \bigcap_{p \in \omega} U_p \neq \emptyset,$$

otherwise β wins. A *tactic* ([10]) for α (resp. for β) is a function $\sigma\colon \mathcal{E}(Y, \mathcal{B}) \to \mathcal{B}$ (resp. $\sigma\colon \mathcal{B} \to \mathcal{E}(Y, \mathcal{B})$) such that $x \in \sigma(x, V) \subset V$ for every $(x, V) \in \mathcal{E}(Y, \mathcal{B})$ (resp. such that for every $U \in \mathcal{B}$, the second component of $\sigma(U)$ is a subset of U). A *winning tactic* (abbr. w.t.) for α (resp. for β) is a tactic σ such that α (resp. β) wins every run of the game compatible with σ, i.e. such that $U_p = \sigma(x_p, V_p)$ (resp. $(x_{p+1}, V_{p+1}) = \sigma(U_P)$) for all $p \in \omega$. We will say that Y is *strongly α-favorable* (resp. *strongly β-favorable*) provided α (resp. β) possesses a winning tactic in the strong Choquet game.

The so-called *Banach–Mazur game* ([10, 38]) is played similarly as the strong Choquet game, except that β chooses only a nonempty $V_p \in \mathcal{B}$ and player α a nonempty $U_p \in \mathcal{B}$, with the same inclusions $V_p \subset U_p \subset V_{p+1}$ and rules as in the strong Choquet game. We can analogously define tactics and winning tactics for α, β, respectively in the Banach–Mazur game. Finally, we will say that Y is *α-favorable* (resp. *β-favorable*) provided α (resp. β) possesses a winning tactic in the Banach–Mazur game.

The notion of ‘winning tactic’ was used by Choquet ([10, Chapter 8]), while Galvin and Telgársky ([21]) uses the term ‘stationary winning strategy’. The reason for this special terminology is that we require that player α remembers only the most recent move of β when deciding what to do next, instead of remembering all the previous moves of β made so far (in this latter case we would say α has a ‘winning strategy’). It is not automatic that an arbitrary winning strategy for α can be reduced to a winning tactic, though for special spaces – including the metrizable ones – this is the case (see [21, Corollary 11] for the Banach–Mazur game, and [10, Theorem 8.7] for the strong Choquet game). Notice that for β the reduction from a winning strategy to a winning tactic is possible in both the Banach–Mazur game and the strong Choquet game (see [21] on this).

For further information about topological games cf. [43] or [27]. Now recall some fundamental results about these games:

THEOREM A. *A topological space Y is not β-favorable iff Y is a Baire space* (*i.e. countable intersections of dense open sets are dense*).

Proof. See [29, Theorem 2] (cf. [38] also) or [22, Theorem 3.16]. □

THEOREM B. *A metric space Y is α-favorable iff Y contains a dense completely metrizable subspace.*

Proof. By [27, Theorem 8.17(i)] Y is α-favorable iff Y is residual in its completion, which in turn is equivalent to having a dense completely metrizable subspace. □

THEOREM C. *A metrizable space Y is not strongly β-favorable iff Y is a hereditarily Baire space* (*i.e. every nonempty closed subspace of Y is a Baire space*).

Proof. See [14, Théorème 4.1(b)] or [43, Theorem 1.3]. □

THEOREM D. *A metrizable space Y is strongly α-favorable iff Y is completely metrizable.*

Proof. See [10, Theorem 8.7 on p. 136]. □

The Banach–Mazur game was first applied in the hyperspace setting by McCoy in [36], which was then generalized by the author in [48, 49]. An application of the strong Choquet game in [50] provided a short proof of the celebrated Beer–Costantini Theorem on Polishness of the Wijsman topology.

THEOREM 4.1. *Suppose that X has property* (P) *and $\mathfrak{D}$ is a weakly Urysohn family. If (X, τ_δ) is strongly α-favorable, then so is $(\mathrm{CL}(X), \tau^*)$.*

Proof. Take a w.t. $\sigma\colon \mathcal{E}(X, \tau_\delta) \to \tau_\delta$ for α in (X, τ_δ). Define a tactic σ^*: $\mathcal{E}(\mathrm{CL}(X), \tau^*) \to \tau^*$ for α as follows: first, for each $V \in \tau_\delta$ and $A \in V^-$ fix a point $x_{A,V} \in A \cap V$. Then given $(A, \mathbf{V}) \in \mathcal{E}(\mathrm{CL}(X), \tau^*)$ where $\mathbf{V} = (V_0, \ldots, V_n)_T \in \mathcal{B}^*$, define

$$\sigma^*(A, \mathbf{V}) = (\sigma(x_{A,V_0}, V_0 \cap M(S)), \ldots, \sigma(x_{A,V_n}, V_n \cap M(S)))_S,$$

where S is obtained from the Urysohnness of $\mathfrak{D}$. Since $A \in S^* \subset T^*$, we have that $A \in \sigma^*(A, \mathbf{V}) \subset \mathbf{V}$. We will show that σ^* is a w.t. for α in $(\mathrm{CL}(X), \tau^*)$.

Suppose that $(A_0, \mathbf{V}_0), \mathbf{U}_0, \ldots, (A_p, \mathbf{V}_p), \mathbf{U}_p, \ldots$ is a run of the strong Choquet game in $(\mathrm{CL}(X), \tau^*)$ such that $\mathbf{U}_p = \sigma^*(A_p, \mathbf{V}_p)$ for all p. Denote $\mathbf{U}_p = (U_0^p, \ldots, U_{n_p}^p)_{S_p}$ and $\mathbf{V}_p = (V_0^p, \ldots, V_{m_p}^p)_{T_p}$ for appropriate $S_p, T_p \in \mathfrak{D}$.

We know that $\mathbf{V}_{p+1} \subset \mathbf{U}_p$ hence by Lemma 3.1 without loss of generality we may assume that $m_{p+1} > n_p = m_p$ and $M(T_{p+1}) \cap V_i^{p+1} \subset M(S_p) \cap U_i^p$ for all $i \leqslant n_p$. Put $n_{-1} = -1$. Then for all $p \in \omega$ and $n_{p-1} < s \leqslant n_p$ we have a run

$$(x_{A_p, V_s^p}, V_s^p \cap M(T_p)), U_s^p \cap M(S_p), \ldots,$$
$$(x_{A_{p+r}, V_s^{p+r}}, V_s^{p+r} \cap M(T_{p+r})), U_s^{p+r} \cap M(S_{p+r}), \ldots$$

of the strong Choquet game in X compatible with σ. Therefore there exists some $x_s \in \bigcap_{r\in\omega} U_s^{p_s+r} \cap M(S_{p_s+r})$ for all $s \in \omega$, where p_s is the unique $p \in \omega$ such that $n_{p-1} < s \leqslant n_p$.

Let $A = \overline{\{x_s : s \in \omega\}}$ and fix $p \in \omega$. It is clear that $x_s \in M(T_{p+1})$ for all $s \in \omega$, hence by (3) $A \in T_p^*$ thus, $A \in \mathbf{V}_p$ for all $p \in \omega$, which means that α wins the run. □

LEMMA 4.1. *Suppose that (X, τ_δ) is regular. Then*

(i) $\hat{X} = \{\overline{\{x\}} : x \in X\}$ *is a closed subset of* $(\mathrm{CL}(X), \tau^*)$*;*

(ii) $\hat{X} \cap (U_0, \ldots, U_n)_S = \hat{X} \cap (M(S) \cap \bigcap_{i\leqslant n} U_i)^- = \hat{X} \cap (M(S) \cap \bigcap_{i\leqslant n} U_i)^+$ *for all* $(U_0, \ldots, U_n)_S \in \mathcal{B}^*$.

Proof. (i) Let $A \in \mathrm{CL}(X) \setminus \hat{X}$. Then there exist $x, y \in A$ with $y \in \overline{\{x\}}^c$. By regularity of X we can find disjoint $U, V \in \tau_\delta$ such that $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$. It is not hard to see now that $A \in U^- \cap V^- \subset \mathrm{CL}(X) \setminus \hat{X}$ thus, $\mathrm{CL}(X) \setminus \hat{X}$ is open in $(\mathrm{CL}(X), \tau^*)$.

(ii) If $\overline{\{x\}} \in \hat{X} \cap (U_0, \dots, U_n)_S$ then by regularity of X and Lemma 2.1(i), $\overline{\{x\}} \subset M(S) \cap \bigcap_{i \leqslant n} U_i$. Conversely, if $\overline{\{x\}} \in \hat{X} \cap (M(S) \cap \bigcap_{i \leqslant n} U_i)^-$ then clearly $\overline{\{x\}} \in \bigcap_{i \leqslant n} U_i^-$, further $x \in M(S)$, and regularity of X with Lemma 2.1(ii) implies that $\overline{\{x\}} \in S^*$. □

We are now prepared to prove the main result of this paper on Polishness of $(\mathrm{CL}(X), \tau^*)$. Note that Polishness of various hypertopologies has already been established using quite different methods; in this respect we should mention [2, 3, 12, 26] and [34] concerning complete metrizability of certain weak hypertopologies, further [16] and [3] about complete metrizability of certain (proximal) hit-and-miss topologies. Our next result characterizes Polishness of hypertopologies with no additional conditions:

THEOREM 4.2. *The following are equivalent:*

(i) $(\mathrm{CL}(X), \tau^*)$ *is Polish;*
(ii) $(\mathrm{CL}(X), \tau^*)$ *is completely metrizable;*
(iii) $(\mathrm{CL}(X), \tau^*)$ *is metrizable and* (X, τ_δ) *is strongly* α*-favorable;*
(iv) $(\mathrm{CL}(X), \tau^*)$ *is* T_1, $\mathfrak{D}$ *contains a countable Urysohn subfamily and* (X, τ_δ) *is strongly* α*-favorable.*

Moreover, if X *is* T_1*, the above are equivalent to*

(v) $(\mathrm{CL}(X), \tau^*)$ *is metrizable and* (X, τ_δ) *is Polish.*

Proof. (iii) ⇒ (ii) Notice that by Corollary 3.1(i), X has property (P) and by Theorem 3.4, $\mathfrak{D}$ is a Urysohn family. Therefore Theorem 4.1 implies that $(\mathrm{CL}(X), \tau^*)$ is strongly α-favorable, which implies (ii) by Theorem D.

(ii) ⇒ (iii) Only strong α-favorability of X needs some comments. It follows from Corollary 3.2 that X is regular so by Lemma 4.1(i), $\hat{X}$ is a completely metrizable subspace of $(\mathrm{CL}(X), \tau^*)$. Consequently, by Theorem D, $\hat{X}$ is strongly α-favorable. Denote $\hat{\mathcal{B}}^* = \{\mathbf{U} \cap \hat{X}\colon \mathbf{U} \in \mathcal{B}^*\}$ and let $\hat{\sigma}\colon \mathcal{E}(\hat{X}, \hat{\mathcal{B}}^*) \to \hat{\mathcal{B}}^*$ be a w.t. for α in the strong Choquet game. Define a tactic $\sigma\colon \mathcal{E}(X, \tau_\delta) \to \tau_\delta$ for α in X as follows: given $(x, U) \in \mathcal{E}(X, \tau_\delta)$ write $\hat{\sigma}(\overline{\{x\}}, \hat{X} \cap U^-) = \hat{X} \cap (U_0, \dots, U_n)_S$ and put $\sigma(x, U) = M(S) \cap \bigcap_{i \leqslant n} U_i$.

Then by Lemma 4.1(ii), $x \in \sigma(x, U) \subset U$. We will show that σ is a w.t. for α in X: let $(x_0, V_0), U_0, \dots, (x_p, V_p), U_p, \dots$ be a run of the strong Choquet game in X such that $U_p = \sigma(x_p, V_p)$ for all $p \in \omega$. Then $(\overline{\{x_0\}}, \hat{X} \cap V_0^-), \hat{X} \cap U_0^-, \dots, (\overline{\{x_p\}}, \hat{X} \cap V_p^-), \hat{X} \cap U_p^-, \dots,$ is a run of the strong Choquet game in $\hat{X}$ compatible with $\hat{\sigma}$, which follows by Lemma 4.1(ii). Accordingly, there exists an $x \in X$ with $\overline{\{x\}} \in \bigcap_{p \in \omega} V_p^-$, whence $x \in \bigcap_{p \in \omega} V_p$ and α wins.

(iii) ⇒ (v) It suffices to show that X is separable and metrizable, since then (v) follows by (iii) and Theorem D. Indeed, in our case X is T_3 by Corollary 3.2

and hence metrizable by Lemma 4.1(i); further, X is separable by Theorem 3.5 and Lemma 4.1(i).

(v) $\Rightarrow$ (iii) follows from Theorem D and the remaining implications are either trivial or follow from Theorem 3.5. □

As for α-favorability of hyperspaces we have:

THEOREM 4.3. *Suppose that X has property* (P) *and* $\mathfrak{D}$ *is a weakly quasi-Urysohn family. If* (X, τ_δ) *is* α*-favorable then so is* $(\mathrm{CL}(X), \tau^*)$*; in particular,* $(\mathrm{CL}(X), \tau^*)$ *is a Baire space.*

Proof. We can adapt the proof of Theorem 4.1 using the Banach–Mazur game instead of the strong Choquet game, using a weakly quasi-Urysohn family instead of a weakly Urysohn family and omitting the x's and A's from the argument. □

Observe, that we cannot use an argument similar to that of in Theorem 4.2 to characterize α-favorability of the (metrizable) hyperspace, since it would require that α-favorability be closed-hereditary, which is not the case (cf. [45, Remark 2.4(2)]). This is not the case in the hyperspace setting either, otherwise α-favorable hyperspaces would be hereditarily Baire; however we have the following

THEOREM 4.4. *There exists a separable, hereditarily Baire,* α*-favorable metric space X such that* $\mathrm{CL}(X)$ *with the Wijsman topology* τ_W *is* α*-favorable but not hereditarily Baire.*

To prove this theorem we need some auxiliary material first. A consequence of a theorem of A. Bouziad ([6, Theorème 2.1]) claims that if there is a perfect mapping from a regular space Y to a metric hereditarily Baire space Z, then Y is a hereditarily Baire space. We have the following variation of this theorem:

THEOREM 4.5. *Let* $f\colon Y \to Z$ *be a perfect mapping from a Tychonoff space Y to a* 1*st countable,* T_3*, hereditarily Baire space* Z*. Then* Y *is a hereditarily Baire space.*

Proof. We can adapt the proof of Theorem 3.7.26 in [18]. In fact, there exists a homeomorphic embedding $g\colon Y \to K$ of Y into a Hausdorff compact space K. Then the diagonal $f \triangle g\colon Y \to Z \times K$ is a perfect mapping (by [18, Theorem 3.7.9]) and a homeomorphic embedding; hence Y is homeomorphic to a closed subspace of $Z \times K$. Now it suffices to use Theorem 12 of [7] claiming, that the product of a 1st countable, regular hereditarily Baire space and a regular compact space is hereditarily Baire. □

Proof of Theorem 4.4. Let X be the separable, hereditarily Baire, α-favorable metric space from [1] having non-hereditarily Baire square. Then by Theorem 4.3, $(\mathrm{CL}(X), \tau_W)$ is α-favorable; further, it is metrizable, since X is separable ([3, Theorem 2.1.5]).

On the other hand $(\mathrm{CL}(X), \tau_W)$ is not hereditarily Baire. Indeed, notice that $Y = X^2 \setminus \{(x, x) : x \in X\}$ is not hereditarily Baire, since $\{(x, x) : x \in X\}$ is hereditarily Baire and X^2 is not. Further, if Z stands for the two-element subsets of X endowed with the relative Wijsman topology, then the natural mapping from Y onto Z is a perfect mapping. Hence by Theorem 4.5, Z is not hereditarily Baire and neither is $(\mathrm{CL}(X), \tau_W)$, since Z is G_δ in $(\mathrm{CL}(X), \tau_W)$ (see [14, Proposition 1.2]). □

5. Applications

As for (strong) α-favorability of hypertopologies we have the following three theorems. The first two extend Theorem 4.3 of [48] and the third one generalizes Theorem 3(ii) of [49]:

THEOREM 5.1. *Suppose that* (X, τ) *is a* (*strongly*) α*-favorable weakly-*R_0 *topological space and* Δ *is a weakly quasi-Urysohn* (*resp. weakly Urysohn*) *family. Then* $(\mathrm{CL}(X), \tau^+)$ *is* (*strongly*) α*-favorable.*

Proof. See Remark 3.1(ii), Remark 3.2(ii) and Theorem 4.1 (resp. Theorem 4.3). □

THEOREM 5.2. *Suppose that* $(X, \mathcal{U})$ *is a* (*strongly*) α*-favorable uniform space and* Δ *is a uniformly weakly quasi-Urysohn* (*resp. uniformly weakly Urysohn*) *family. Then* $(\mathrm{CL}(X), \tau^{++})$ *is* (*strongly*) α*-favorable.*

Proof. See Remark 3.1(ii), Remark 3.2(i) and Theorem 4.1 (resp. Theorem 4.3). □

THEOREM 5.3. *Suppose that* (X, d) *is a* (*strongly*) α*-favorable metric space and* Δ_1 *contains the singletons. Then* $(\mathrm{CL}(X), \tau_{\mathrm{weak}})$ *is* (*strongly*) α*-favorable.*

Proof. See Remark 3.1(iii), Remark 3.2(i) and Theorem 4.1 (resp. Theorem 4.3). □

THEOREM 5.4. *Let* (X, τ) (*resp.* $(X, \mathcal{U})$) *be a topological* (*resp. uniform*) *space and* $\Delta \subset \mathrm{CL}(X)$. *The following are equivalent:*

(i) $(\mathrm{CL}(X), \tau^+)$ (*resp.* $(\mathrm{CL}(X), \tau^{++})$) *is Polish;*
(ii) $(\mathrm{CL}(X), \tau^+)$ (*resp.* $(\mathrm{CL}(X), \tau^{++})$) *is completely metrizable;*
(iii) $(\mathrm{CL}(X), \tau^+)$ (*resp.* $(\mathrm{CL}(X), \tau^{++})$) *is metrizable and* (X, τ) (*resp.* $(X, \mathcal{U})$) *is strongly* α*-favorable;*
(iv) $(\mathrm{CL}(X), \tau^+)$ (*resp.* $(\mathrm{CL}(X), \tau^{++})$) *is* T_1, Δ *contains a countable* (*uniformly*) *Urysohn subfamily and* (X, τ) (*resp.* $(X, \mathcal{U})$) *is strongly* α*-favorable.*

Moreover, if (X, τ) (*resp.* $(X, \mathcal{U})$) *is* T_1, *the above are equivalent to*

(v) $(\mathrm{CL}(X), \tau^+)$ (*resp.* $(\mathrm{CL}(X), \tau^{++})$) *is metrizable and* (X, τ) (*resp.* $(X, \mathcal{U})$) *is Polish.*

Proof. See Theorem 4.2. □

The following theorem generalizes Theorem 3.6. and Theorem 4.2 of [26]:

THEOREM 5.5. *Let (X, d) be a metric space and Δ_1 contain the singletons. The following are equivalent:*

(i) $(\mathrm{CL}(X), \tau_{\mathrm{weak}})$ *is Polish;*
(ii) $(\mathrm{CL}(X), \tau_{\mathrm{weak}})$ *is completely metrizable;*
(iii) $(\mathrm{CL}(X), \tau_{\mathrm{weak}})$ *is 2nd countable and (X, d) is completely metrizable;*
(iv) $(\mathrm{CL}(X), \tau_{\mathrm{weak}})$ *is 2nd countable and (X, d) is Polish.*

Proof. Observe that if Δ_1 contains the singletons, then $(\mathrm{CL}(X), \tau_{\mathrm{weak}})$ is T_1 and by Remark 2.1(iii) it is completely regular as an initial topology; hence the theorem follows from Theorem 3.5 and Theorem 4.2. □

Note that $H(A, B) = \max\{e(A, B), e(B, A)\}$ is the so-called *Hausdorff metric* between $A, B \in \mathrm{CL}(X)$. Confer Theorem 3.3 and Theorem 4.3 of [26] with:

COROLLARY 5.1. *Let (X, d) be a completely metrizable space. Let Δ_1 and Δ_2, respectively be separable with respect to the induced Hausdorff metric and Δ_1 contain the singletons.*

Then $(\mathrm{CL}(X), \tau_{\mathrm{weak}})$ is a Polish space.

Proof. Let $\Omega_i \subset \Delta_i$ be countable and dense in Δ_i with respect to H for each $i = 1, 2$. Since Δ_1 contains the singletons, it is easy to show that X is separable; let $\mathcal{B}$ be a countable base of X. Then the collection of all

$$\bigcap_{i \leqslant n} U_i^- \cap \bigcap_{j \leqslant m} (D^{\leftarrow}(\cdot, A_j) \cap e^{\leftarrow}(\cdot, B_j))$$

with $U_0, \cdots, U_n \in \mathcal{B}$, $A_j \in \Omega_1$ and $B_j \in \Omega_2$ for all $j \leqslant m$ and $m, n \in \omega$, forms a countable base for τ_{weak} and Theorem 5.5 applies. □

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